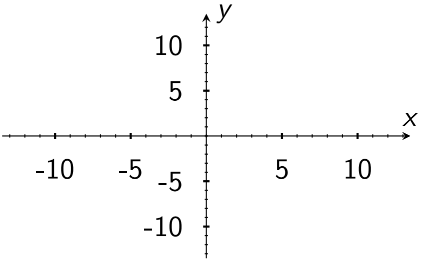
Lecture 01
12.1/12.2 The geometry of space and vectors

Jeremiah Southwick

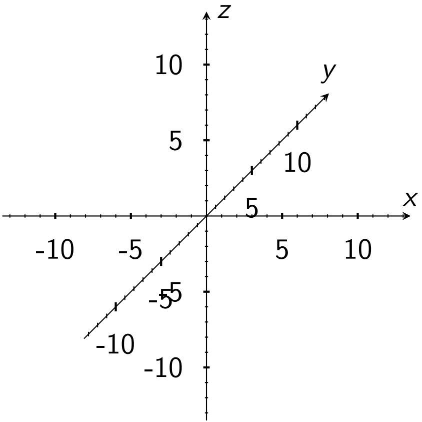
January 14, 2019

§12.1 Plane vs. Space

Plane



Space



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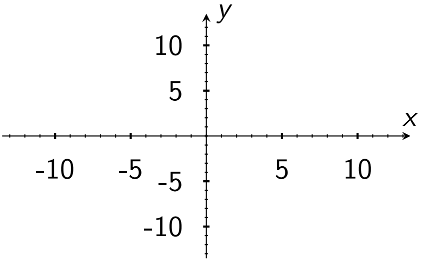

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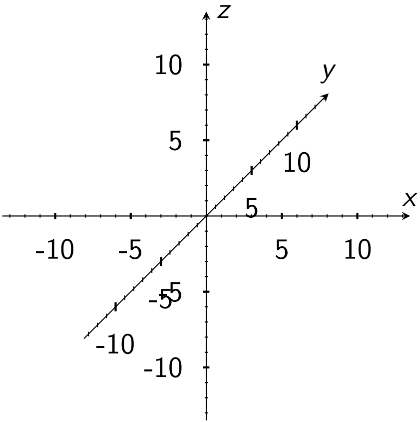

To represent three dimensions, we add a z-axis.

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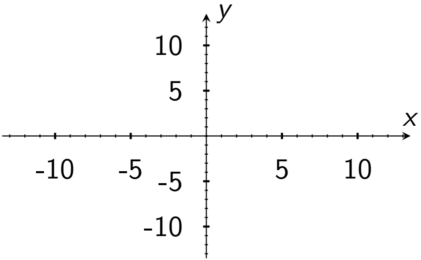

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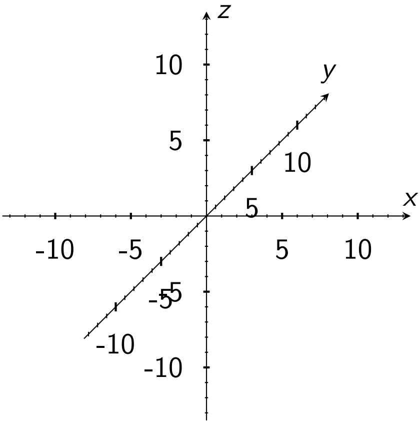

To represent three dimensions, we add a z-axis.
We always draw the coordinate axes using the right-hand rule.

Plane vs. Space

Since we have three variables, points are represented as triplets.

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(x, y)	(x, y, z)

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$x^2 + y^2 = 4$	$x^2 + y^2 + z^2 = 9$

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We can use online tools to visualize these objects.
`https://www.geogebra.org/3d?lang=en`

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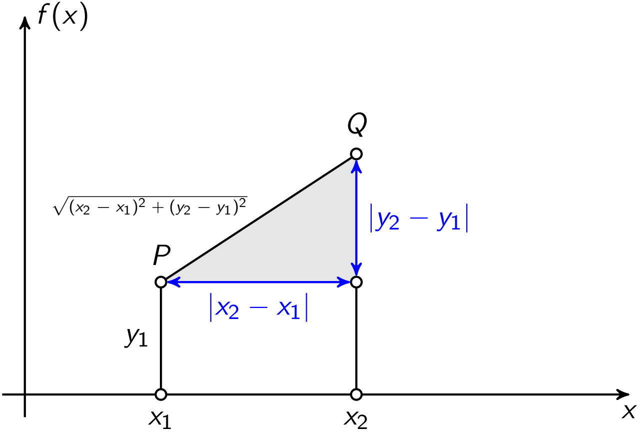

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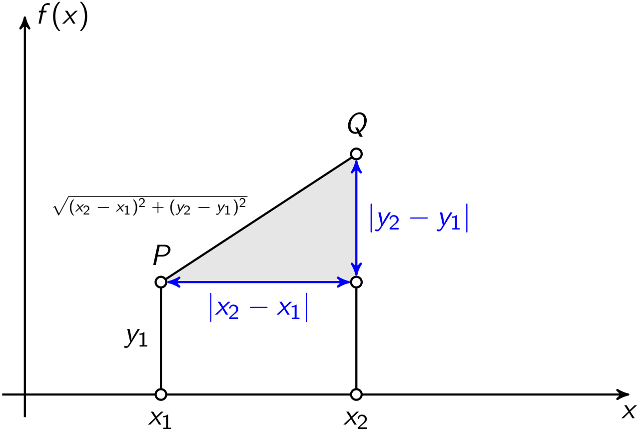


A similar formula holds in 3D (and in higher dimensions).

DISTANCE!

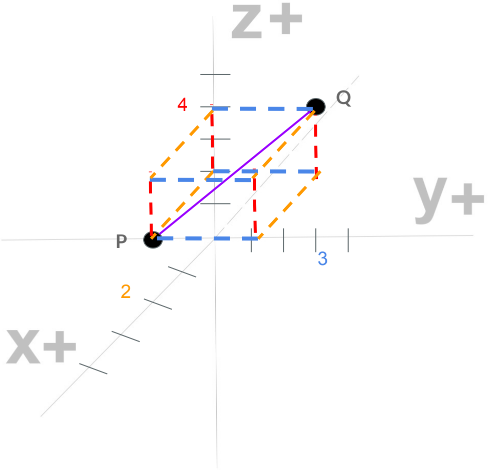
How do we find the distance between two points?

|PQ|=?

Let's try an example:

P=(2,0,2)

Q=(0,3,4)



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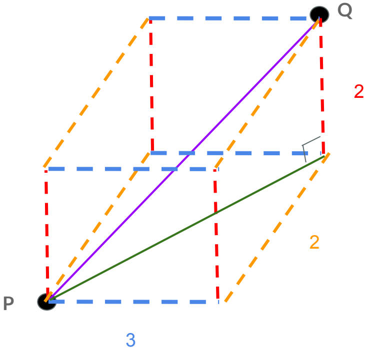

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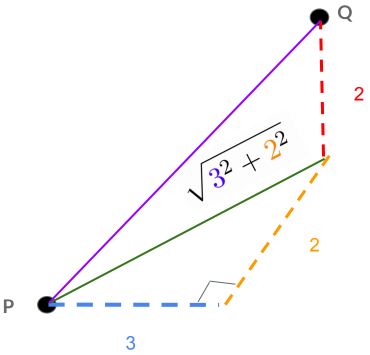

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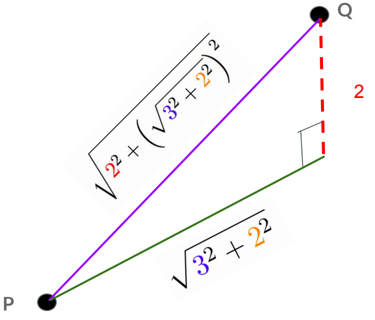

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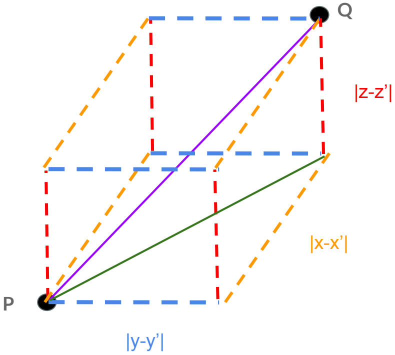

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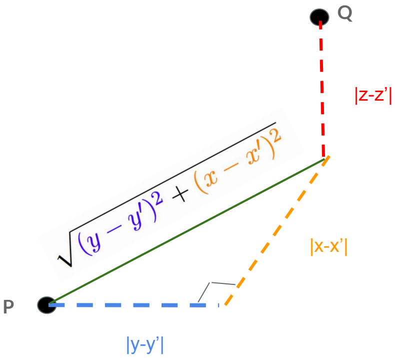

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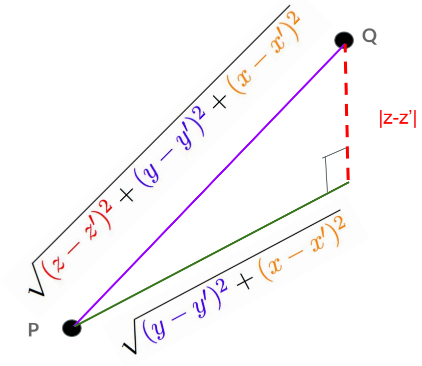


Distance and Spheres

Thus we have the following distance formula.

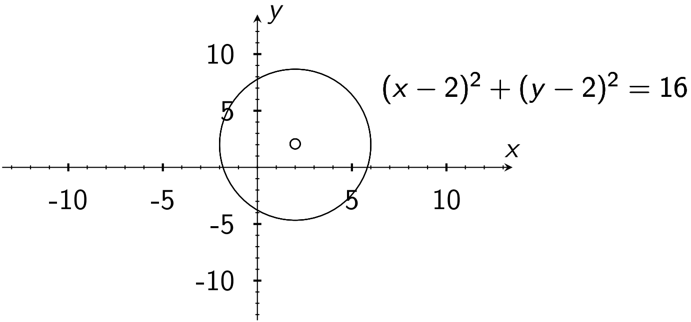
Definition

The distance between two points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *is*

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Recall Circles

In the plane, a circle is all the points a fixed distance away from a fixed point.



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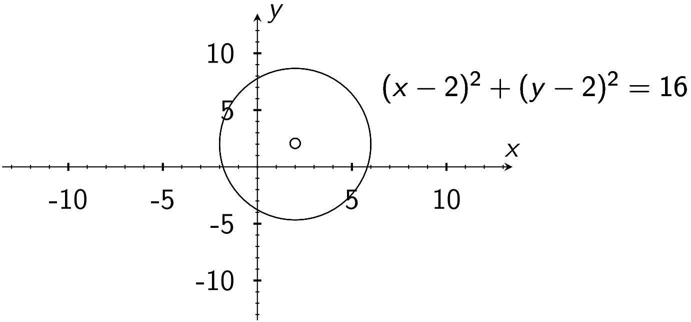
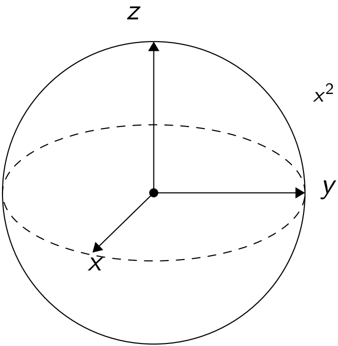


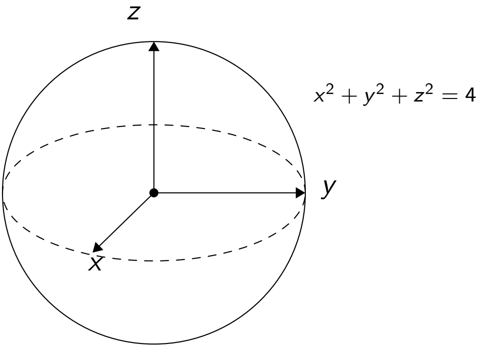
In space, the set of all points a fixed distance from a fixed point is a sphere.

Spheres



$$x^2 + y^2 + z^2 = 4$$

Spheres



Using the distance formula where we make (x, y, z) a point on the sphere with center (x_0, y_0, z_0) and radius r, we get the standard equation for a sphere.

Spheres

If (x, y, z) has distance r from the center (x_0, y_0, z_0), then the following equation holds:

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r.$$

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Definition

The standard equation of a sphere with center (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Spheres

Example

Find the center and radius of $x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$

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We complete the square in each variable, adding the same amount to both sides of the equation:

$$\left[x^2+3x+\left(\frac{3}{2}\right)^2\right]+y^2+\left[z^2-4z+\left(-\frac{4}{2}\right)^2\right] = -1+\left(\frac{3}{2}\right)^2+\left(-\frac{4}{2}\right)^2$$

This simplifies to

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4}.$$

Thus the sphere has center $\left(-\frac{3}{2}, 0, 2\right)$ and radius $\sqrt{21/4}$.

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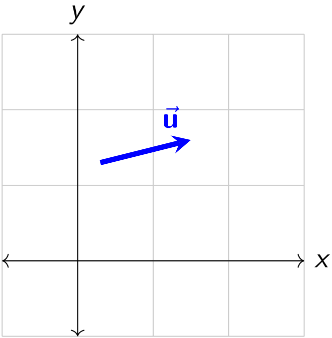
Put another way, a vector is the following information: A direction and a length/magnitude.



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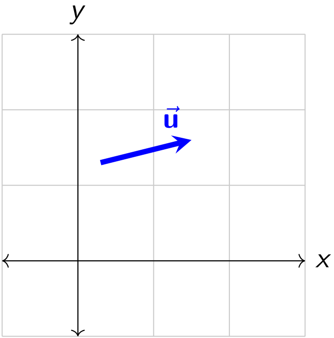

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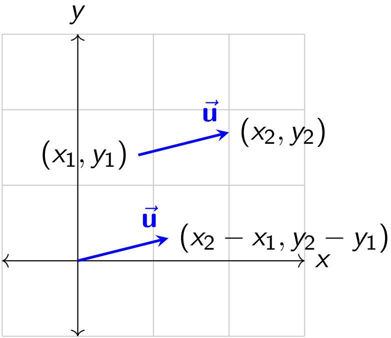
A vector models any application where force is involved: velocity, displacement, work, etc.



Since it doesn't matter where we draw a vector, we will usually place the initial point at the origin. This is called *standard position*.

Standard Position

Definition

If a vector $\vec{\mathbf{v}}$ *goes from* (x_1, y_1, z_1) *to* (x_2, y_2, z_2)*, then the same vector in standard position goes from* $(0, 0, 0)$ *to* $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$*. In this case we write*

$$\vec{\mathbf{v}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

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$$\langle u_1, u_2, u_3 \rangle = \langle v_1, v_2, v_3 \rangle \iff u_1 = v_1, u_2 = v_2, \text{ and } u_3 = v_3$$

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How would we write down the vector starting at $(-7, 5, 0)$ *and going to* $(3, -1, 4)$*?*

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$\vec{\mathbf{v}} = \langle 10, -6, 4 \rangle$.

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Example

The length of $\langle 7, 3, -2 \rangle$ *is* $\sqrt{49 + 9 + 4} = \sqrt{62}$.

Zero Vector

The only vector with length 0 is the vector that starts at a point and ends at the same point. We have special notation for this vector.

Definition

The vector of all zeros is denoted $\vec{\mathbf{0}} := \langle 0, 0, 0 \rangle$*. This notation can be used no matter what dimension, i.e., in 2D we have* $\vec{\mathbf{0}} = \langle 0, 0 \rangle$*.*